

Math 221  
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December 9, 2010

### Navier-Stokes Equations Applied to Fluid Mechanics

In order to convey the importance of the Navier-Stokes equations in fluid mechanics, a quick explanation of fluid mechanics has to be discussed. A fluid, by definition, is any material continuum that cannot endure a static shear stress; fluids respond to a shear stress with an irrecoverable flow. Therefore, gases and liquids are types of fluids. Liquids are normally considered to be incompressible while gases are considered to be compressible ("Fluid Mechanics: Overview"). Even a solid can be considered to be a fluid if it is broken into several small pieces. This is seen as quick sand exhibits fluid characteristic by the fact that it can drown people. However, fluids are most commonly liquids and gases; to have a solid act as a fluid is rare (Merriam).

With a sense of fluids established, fluid mechanics is a branch of continuum mechanics. Fluid mechanics is used to describe how fluids flow. The field consists of fluids under static and dynamic conditions. Several aspects of fluid mechanics include static or dynamic flow, compressible or incompressible flow, ordinary or turbulent flow, and single phase or multi phase flow. All of these different flows are modeled by a general set of equations known as the Navier-Stokes equations which is very helpful in fluid mechanics (Merriam).

As the name implies, the Navier-Stokes equations were named after Claude-Louis Navier and George Gabriel Stokes ("Fluid Mechanics: Navier-Stokes Equations"). The equations were developed by Navier in 1831 and more vigorously by Stokes in 1845 (*Derivation of the Navier-Stokes Equations and Solutions*). Navier used a molecular point of view to discover the

equations while Stokes used a continuous point of view (Merriam). Navier derived the Navier-Stokes equation as an expansion of Euler's equation in order to include viscosity. Poisson generalized Navier's equations to compressible fluids, and De Saint-Venant formulated fully continuous derivations. Stokes independently arrived at the results of Poisson and Saint-Venant using a continuous model ("Navier-Stokes Equation"). These equations are used to describe the motion of fluid as continuum substances. The basis for these equations is that changes in fluid momentum is a result of body forces and the external forces that are applied to the fluid. The Navier-Stokes equations hold true for any sum of forces at any region of the fluid (Merriam).

The Navier-Stokes equations are derived from the basic principles of conservation of mass, momentum, and energy ("Derivation of the Navier-Stokes Equations"). Conservation of mass can be stated by the following: 'the rate at which mass increases within the control volume = the rate at which mass enters the control volume through its four boundaries.' Assuming that the control volume, or CV, is an area  $x$ -long and  $y$ -high, the conservation of mass can be derived. With  $\rho$  being the average density of the fluid within the control volume, the mass within the control volume equals  $(\rho \cdot \Delta x \cdot \Delta y)$ , and the time rate of change of mass within the CV

is  $\frac{\partial(\rho \cdot \Delta x \cdot \Delta y)}{\partial t} = (\Delta x \cdot \Delta y) \frac{\partial \rho}{\partial t}$ . Next, the rate at which mass enters through the boundaries of the

CV has to be considered. There are four sides of the CV with  $x$  denoting the horizontal axis and  $y$  denoting the vertical axis. The rates are as follows with the subscripts denoting a particular side of the CV,  $u$  denoting velocity along the  $x$ -axis and  $v$  denoting velocity along the  $y$ -axis:

$(\rho \cdot u \cdot \Delta y)_1, (-\rho \cdot u \cdot \Delta y)_2, (\rho \cdot v \cdot \Delta x)_3, (-\rho \cdot v \cdot \Delta x)_4$ . Adding these contributions and equating the result to

the time rate of change of mass within the CV yields the following:  $\Delta x \Delta y \frac{\partial \rho}{\partial t} = [(\rho u \Delta y)_1 -$

$(\rho u \Delta y)_2] + [(\rho v \Delta x)_3 - (\rho v \Delta x)_4]$ . This can be rewritten as  $\frac{\partial \rho}{\partial t} = \frac{[(\rho u)_1 - (\rho u)_2]}{\Delta x} + \frac{[(\rho v)_3 - (\rho v)_4]}{\Delta y}$ .

From calculus,  $\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$  and  $\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$ . By applying these limits to  $\frac{\partial \rho}{\partial t} = \frac{[(\rho u)_1 - (\rho u)_2]}{\Delta x} + \frac{[(\rho v)_3 - (\rho v)_4]}{\Delta y}$  and bringing all of the terms to the left hand side,  $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$  is obtained. This is the first Navier-Stokes equation (*Derivation of the Navier-Stokes Equations and Solutions*).

The second and third equations are derived from momentum conservation. There are two directions, x and y, that velocity can occur, and therefore, there are two momentum equations. The conservation of momentum can be stated as 'the rate of change of u-momentum within a control volume is equal to the net rate at which u-momentum enters the control volume + Forces (pressure, viscous and body) acting on the control volume in the x-direction.' The same can be stated about v-momentum in the y-direction. By using a CV with dimensions x by y as in the conservation of mass equation, the momentum equations can be derived. The rate of change of u-momentum within the CV =  $\Delta x \Delta y \frac{\partial(\rho u)}{\partial t}$ , and u-momentum can enter the CV from four directions. Again, with x denoting the horizontal, y denoting the vertical, and the number subscript denoting the face, the rates of which u-momentum enters the CV are  $(\rho u^2 \Delta y)_1$ ,  $(-\rho u^2 \Delta y)_2$ ,  $(\rho uv \Delta x)_3$ ,  $(-\rho uv \Delta x)_4$ . There are pressure forces that act on face 1 and 2 that are  $(P \Delta y)_1$  and  $(-P \Delta y)_2$  respectively. Note that  $\rho$  stands for density while P stands for pressure. Taking  $\tau$  to represent a viscous force acting on the fluid, there are four different viscous forces that act on the CV. The two normal viscous forces on face 1 are  $(-\tau_{xx} \Delta y)_1$  and  $(\tau_{yx} \Delta x)_1$ . The normal viscous force on face 2 is  $(\tau_{xx} \Delta y)_2$ , and the tangential viscous force on face 3 is  $(-\tau_{yx} \Delta x)_3$ . By summing up the momentum, pressure, and viscous forces contributions, dividing through by  $\Delta x \Delta y$ , and taking the limits as  $\Delta x$  and  $\Delta y$  go to zero, the second equation is derived:  $\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + P)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$ . The derivation of the v-momentum equation is done in the same manner as the u-

momentum equation and leads to  $\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2 + P)}{\partial y} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$  being the third equation (*Derivation of the Navier-Stokes Equations and Solutions*).

The fourth and final Navier-Stokes equation is the energy equation. This equation is basically a generalized form of the first law of Thermodynamics. The first law of Thermodynamics states that the rate at which the total energy increases within a CV is equal to the rate at which total energy enters the CV + the rate at which work is done on the CV boundary by surface forces + the rate at which heat is added to the CV at the surfaces by heat conduction. With the specific total energy as E, the rate at which the total energy increases within the CV  $= \frac{\partial(\rho E)}{\partial t} \Delta x \Delta y$ . The rate at which total energy enters the CV through the four faces can be expressed as

$$\rho u E \Delta y_1 - \rho u E \Delta y_2 + \rho v E \Delta x_3 - \rho v E \Delta x_4 = \Delta x \Delta y \left[ \frac{\rho u E_1 - \rho u E_2}{\Delta x} \right] + \Delta x \Delta y \left[ \frac{\rho v E_3 - \rho v E_4}{\Delta y} \right];$$

the terms in the square brackets go to  $\frac{\partial(\rho u E)}{\partial x}$  and  $\frac{\partial(\rho v E)}{\partial y}$  as  $\Delta x$  and  $\Delta y$  go to zero. Pressure and viscous forces do work on the fluid, and their contributions through the four faces can be expressed as

$$(P u - \tau_{xx} u - \tau_{xy} v) \Delta y_1 - (P u - \tau_{xx} u - \tau_{xy} v) \Delta y_2 + (P v - \tau_{yx} u - \tau_{yy} v) \Delta x_3 - (P v - \tau_{yx} u - \tau_{yy} v) \Delta x_4 =$$

$$\Delta x \Delta y \left[ \frac{(P u - \tau_{xx} u - \tau_{xy} v)_1 - (P u - \tau_{xx} u - \tau_{xy} v)_2}{\Delta x} \right] + \Delta x \Delta y \left[ \frac{(P v - \tau_{yx} u - \tau_{yy} v)_3 - (P v - \tau_{yx} u - \tau_{yy} v)_4}{\Delta y} \right] \approx$$

$$-\Delta x \Delta y \frac{\partial(P u - \tau_{xx} u - \tau_{xy} v)}{\partial x} - \Delta x \Delta y \frac{\partial(P v - \tau_{yx} u - \tau_{yy} v)}{\partial y}. \text{ With } k \text{ being conductivity and } T \text{ being}$$

temperature, the rate at which heat is added to the CV at the surfaces by heat conduction is expressed as

$$-k \frac{\partial T}{\partial x} \Delta y_1 + k \frac{\partial T}{\partial x} \Delta y_2 - k \frac{\partial T}{\partial y} \Delta x_3 + k \frac{\partial T}{\partial y} \Delta x_4 = \Delta x \Delta y \left[ \frac{k \frac{\partial T}{\partial x}_2 - k \frac{\partial T}{\partial x}_1}{\Delta x} \right] + \Delta x \Delta y \left[ \frac{k \frac{\partial T}{\partial y}_4 - k \frac{\partial T}{\partial y}_3}{\Delta y} \right] \approx$$

$\Delta x \Delta y \frac{\partial(k \frac{\partial T}{\partial x})}{\partial x} + \Delta x \Delta y \frac{\partial(k \frac{\partial T}{\partial y})}{\partial y}$ . By summing up all the parts and dropping the common factor  $\Delta x \Delta y$ , the energy equation becomes  $\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho u E)}{\partial x} + \frac{\partial(\rho v E)}{\partial y} + \frac{\partial(uP)}{\partial x} + \frac{\partial(vP)}{\partial y} = \frac{\partial(u\tau_{xx} + v\tau_{xy})}{\partial x} + \frac{\partial(u\tau_{yx} + v\tau_{yy})}{\partial y} + \frac{\partial(k \frac{\partial T}{\partial x})}{\partial x} + \frac{\partial(k \frac{\partial T}{\partial y})}{\partial y}$  (Derivation of the Navier-Stokes Equations and Solutions).

With the Navier-Stokes equations derived, cases where they are used in fluid dynamics can be viewed. Seeing as the Navier-Stokes equations are quite complex, "only the simplest cases can be solved" with the help of calculus. These simple cases generally involve non-turbulent, steady flow that does not change with time ("Fluid Mechanics: Navier-Stokes Equations"). The first simple case to be viewed is a steady flow in a pipe. The second Navier-Stokes equation can be written as  $\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla P + F + \frac{\mu}{\rho} \nabla^2 u$ , where  $u$  = velocity vector field,  $P$  = pressure,  $\rho$  = density,  $\mu$  = viscosity,  $F$  = external force per unit mass,  $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$ , and  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ . The left-hand side of this equation becomes zero for steady flow, and if there is no external force, the equation can be written as follows in cylindrical coordinates:  $0 = -\frac{dp}{dz} + \mu \frac{1}{r} \frac{d}{dr} \left( r \frac{dv}{dr} \right)$ , where  $v$  is the velocity in the  $z$  direction. With a constant pressure drop,  $\frac{dp}{dz}$  becomes a negative constant and the solution becomes  $v(r) = v_m \left( \frac{1-r^2}{R^2} \right)$ , where  $v_m = \frac{R^2}{4\mu} \left( \frac{-dp}{dz} \right)$  is the velocity at the center, and  $R$  is the radius of the pipe. The boundary conditions for this case are  $v = v_m$  at  $r = 0$  and  $v = 0$  at  $r = R$ . Another simple case has to do with hydrostatics. Hydrostatics has to do with fluids at rest, and for this special case,  $u = 0$  in the Navier-Stokes equations, the external force  $F = -g$  with  $g$  being the force of gravity, and the pressure gradient becomes  $dp$ .  $\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla P + F + \frac{\mu}{\rho} \nabla^2 u$  becomes  $dp = -\rho g dy$ . By

taking two pressure points  $p_2$  and  $p_1$ , this simple equation becomes  $p_2 - p_1 = -\rho g(y_2 - y_1)$ . This equation shows how pressure becomes less with higher elevation (*Fluid Dynamics and the Navier-Stokes Equations*).

Although there are other special cases in which the Navier-Stokes equations can be solved, a majority of the solutions of the Navier-Stokes equations can only be found with the aid of computers. There is a field of sciences called computational fluid dynamics that specializes in using the Navier-Stokes equations to solve complex situation (“Fluid Mechanics: Navier-Stokes Equations”). Within computational fluid mechanics, there are several methods developed such as finite difference, finite volume, finite element, boundary element, and second order methods such as Godunov’s numerical scheme. “The finite element method is based on Taylor series expansion around node points (grid) in the field.” By looking at the volume around the node and forcing a quantity such as momentum to be balanced around every node volume, finite volume is established (Merriam). Computational fluid dynamics continue to progress as better methods can be devised using the Navier-Stokes equations.

When used properly, the Navier-Stokes equations can be used to describe the motion of fluid substances. As an overview, these “equations state that changes in momentum force of fluid particles depend only on the external pressure and internal viscous forces similar to friction acting on the fluid” (“Fluid Mechanics: Navier-Stokes Equations”). Therefore, these equations can be used to model various forms of fluids and determine their behaviors. The Navier-Stokes equations offer potential for vast discoveries within fluid mechanics.

## Works Cited

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