A

Modeling Motion of Paper Plane

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Abstract

Differential equation provide a model of how paper plane fly under the curtained condition; this can provide a best angle for throwing off.

1. Introduction

When this project first assigned, I had difficulty coming up with any ideas for a topic. I wanted to choose a subject that was challenging, but not something that was beyond my abilities to study. I also thought it was important to choose a topic that I was genuinely interested in and wanted to research. This project comes out with my childhood dream—throwing out the furthest paper plane. The classic physics told me that the paper plane cannot fly across the city because of limitation of material and initial velocity that was given by arm. However, this project is helping me to find out how can I make a best shoot under the certain condition.

Firstly, there are several assumptions for the airplane model.

1) The paper airplane has the best balance to make it fly roughly though a parabola. The paper airplane will not make any shape change during the flight.
2) The air speeding is at a constant value, which will not change before paper plane landed.

With the assumption above we can created a planer coordinate with x-axis and y-axis for vertical and horizontal displacement. And we also need separate the force into vertical and horizontal. With x and y indicts the displacement, x' and y' would be the velocity of those two direction, also x'' and y'' would be the acceleration.

According to the Stokes' law: exerted on objects with very small Reynolds numbers (e.g., very small particles) in a continuous viscous fluid. Stokes' law is derived by solving the Stokes flow limit for small Reynolds numbers of the generally unsolvable Navier–Stokes equations:

$$F_d = 6\pi\eta RV \quad [F1]$$

F_d is the frictional force acting on the interface between the fluid and the particle (in N),

η is the fluid's viscosity (in [kg m-1 s-1], 1.78×10^{-5}in this case),

R is the radius of the spherical object (in m), and

V is the particle's velocity (in m/s).

The paper plane has the face area approximately 1/4 of an A4 paper which is $155.925cm^2$ and 0.01mm thick. So we can get R by calculate the volume and transfer it to spherical:

$$\frac{4}{3}\pi R^3 = 0.0155925m^2 * 0.000001m$$

So we get R=0.003339m

So the air resistances of two directions are:

$$F_{dx} = 6\pi * 1.78 * 10^{-5} * 0.0015498 * X'$$

$$= 1.12 * 10^{-6}x'$$

And also, $F_{dy} = 1.12 * 10^{-6}y'$

The weight of the paper plane is a very important value for this mathematical model. I used the standard weight 70g per meter square. According to the area of paper plane the weight would be:

$$m = \frac{0.0155925m^2}{1m^2} * 70g = 1.091475g = 1.09 * 10^{-3}kg$$

So far, we can conclude the force equation on the vertical direction according to the classical physics:

$$my'' + F_{dy} + mg = 0$$

$$i.e.\ \ 1.09 * 10^{-3}y'' + 1.12 * 10^{-6}y' + mg = 0\ \ [1]$$

The differential equation above is a nonhomogeneous equation with constant coefficients. We solve the homogeneous equation first. We assume that $y = e^{rt}$, and it then follows that r must be a root of the characteristic equation.

$$1.09 * 10^{-3}r^2 + 1.12 * 10^{-6}r = 0$$

The possible values of r are $r = 0\ and\ r = -1.03 * 10^{-3}$, so the general solution can be got as:

$$y = c_1e^{-0t} + c_2e^{-0.00103t}\ [2]$$

And now, we assume the particular solution $Y(t) = At$, so that $Y'(t) = A\ and\ Y''(t) = 0$. Take those back to [1], we get:

$$1.12 * 10^{-6} * A + mg = 0$$

$$so, A = -\frac{mg}{1.12 * 10^{-6}} = -2035.38$$

$$Y(t) = -9537.5t$$

So the solution becomes:

$$y = c_1e^{-0t} + c_2e^{-0.00103t} - 9537.5t \ \ [3]$$

The paper plane would though my head, so we can assume the initial condition $y(0) = 1.75m$. Put this value into the equation above we can get $c_1 + c_2 = 1.75$. To determine c_2 we need to know the initial velocity for the paper plane which is the speed of my arm motion. It is hard to test how fast I can throw out the air plane; however, there is one man to the same test for me (http://www.paperplane.org). He confidently estimates the plane leaves my hand at 60 miles per hour. This value is base on his baseball pitches clocked and his calculation. In this model I will use this data as mine. So we can get

$$60\frac{miles}{hour} = 96560.34\frac{meters}{hout} = 26.82\frac{meter}{second}$$

This is the initial velocity for the paper plane, but not the initial velocity for the vertical direction. Because of one goal of this differential equation model is to determine the best angle for throwing the paper plane. We will leave the angle as an undetermined character: Θ, which is the angle between initial velocity and ground. And so the vertical initial velocity will be:

$$y'(0) = 26.82sin\theta$$

From [1], we can get:

$$y' = -0.00103c_2e^{-0.00103t} - 9537.5$$

Take in the initial condition we get:

$$26.82sin\theta = -0.00103c_2 - 9537.5$$

$$i.e.\ c_2 = -9.26 * 10^6 - 2.6 * 10^4 sin\theta$$

And so $c_1 = 1.75 + 9.26 * 10^6 + 2.6 * 10^4 sin\theta$

The [1] is now become:

$$y = 1.75 + 9.26 * 10^6 + 2.6 * 10^4 sin\theta$$

$$- (9.26 * 10^6 + 2.6 * 10^4 sin\theta)e^{-0.00103t}$$

$$- 9537.5t \ \ [4]$$

Now, let's back to the horizontal direction. The horizontal direction involved wind speed which provides the paper air planes a extra force. The data 1mph of wind speed is the current wind speed at Lawrence, KS 2010-12-16 22:11. Using the [F1] formula above, we can get:

$$F_w = 6\pi\eta RV = 0.001575\ N$$

So now the equation for horizontal motion can be described as classical physics:

$$mx'' + F_{dx} + F_w = 0\ \ [5]$$

$$i.e.\ 1.09 * 10^{-3}x'' + 1.12 * 10^{-6}x' + 1.58 * 10^{-3} = 0$$

Again, we solved the homogenous equation first, since the first two coefficients are as same as the vertical direction, we can directly get the solution for homogenous equation:

$$x = c_1e^{-0t} + c_2e^{-0.00103t}$$

Then we assume the particular solution is $X(t) = At$, so $X'(t) = A\ and\ X''(t) = 0$, taken those back to the [5], we get:

$$1.12 * 10^{-6}A + 1.58 * 10^{-3} = 0$$

$$i.e.\ A = 1410.71$$

So the general solution becomes:

$$x = c_1e^{-0t} + c_2e^{-0.00103t} + 1410.71t$$

Like the vertical direction, this initial displacement would be 0, and initial velocity would be $26.82cos\theta$.

$$x(0) = c_1 + c_2 = 0$$

$$x'(0) = -0.00103c_2 + 1410.71 = 26.82cos\theta$$

$$i.e.\ c_2 = 1.37 * 10^6 - 2.6 * 10^4cos\theta$$

$$c_1 = -1.37 * 10^6 + 2.6 * 10^4cos\theta$$

The solution finally becomes:

$$x = -1.37 * 10^6 + 2.6 * 10^4cos\theta$$

$$+ (1.37 * 10^6 - 2.6 * 10^4cos\theta)e^{-0.00103t}$$

$$+ 1410.71t\ \ [6]$$

2. Results

The motion description function had been already concluded:

$$x = -1.37 * 10^6 + 2.6 * 10^4 cos + (1.37 * 10^6 - 2.6 * 10^4 cos\theta)e^{-0.00103t} + 1410.71t$$

$$y = 1.75 + 9.26 * 10^6 + 2.6 * 10^4 sin\theta - (9.26 * 10^6 + 2.6 * 10^4 sin\theta)e^{-0.00103t} - 9537.5t$$

The most accurate two solve these functions is to use $sin\theta$ to represent t from y function than take into the x function to calculate for the maximum of x. However, those two functions involved two variables and $e^{-0.00103t}$ and $1410.71t$ are all in the function, so that it is out of my ability to solved those function, I tried Taylor's series to approximate $sin\theta$ or $cos\theta$, but didn't work. Instead, I use different value of θ to get the table of x and y; and put x and y into one chart, get the motion graph. The graph is shown below, with different θ, there is not much change on graph.

Graph.1 Sample Graph of Motion Description

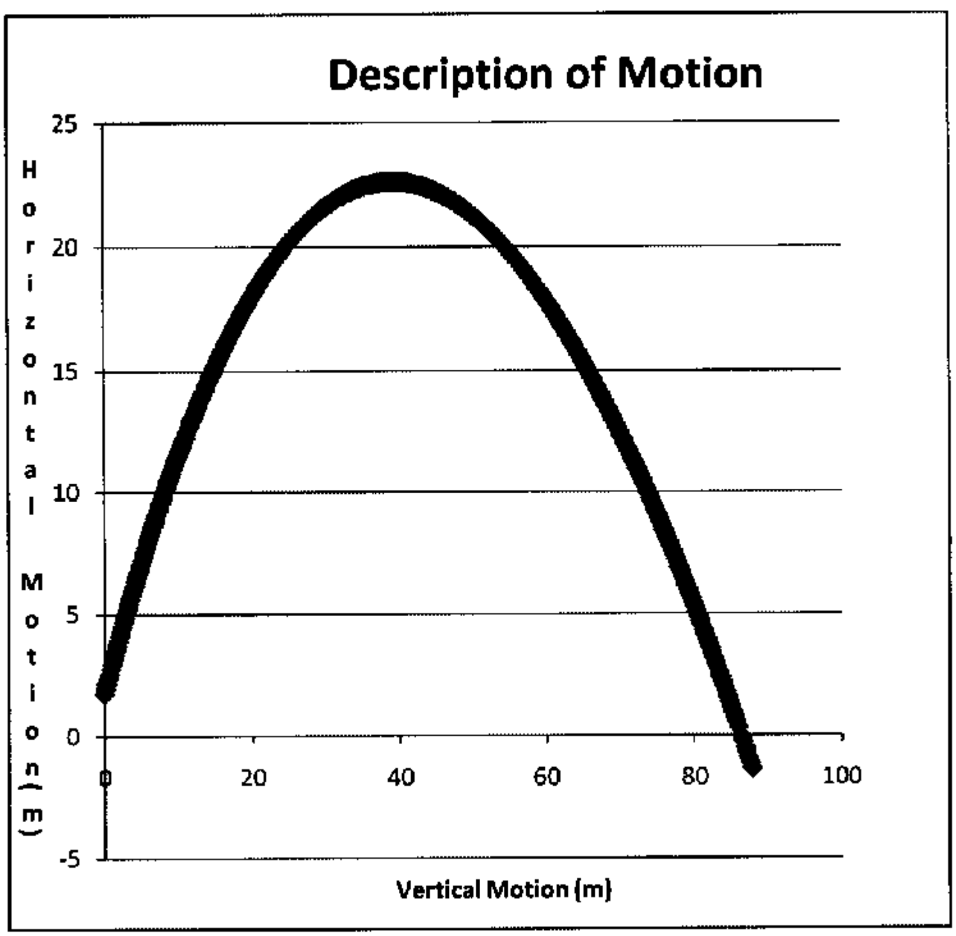

Table.1 Sample Table of Motion

Time(s)	Vertical Displacement(m)	Horizontal Displacement(m)
3.97	0.894351	84.86538
3.98	0.696842	85.1076
3.99	0.498354	85.34997
4	0.298885	85.59248
4.01	0.098435	85.83513
4.02	-0.10299	86.07793

The Graph and table above is from the $\theta = 45°$, it is easily can see that at about 4.01s the paper plane landed, so the Horizontal Displacement 85.83513m is the maximum distance that paper plane can go.

As the same approach above, I did multiply tests for different θ values, the data is shown below:

θ(°)	Flying Time(s)	Maximum Distance(m)
40	3.67	83.45282
41	3.74	84.13196
42	3.81	84.67518
43	3.88	85.18058
44	3.95	85.64816
45	4.01	85.83513
46	4.07	85.98884
47	4.14	86.23677
48	4.2	86.20779
49	4.27	86.37512
50	4.32	86.04046
60	4.86	80.10836

It is easily can conclude that the paper plane can go as far as **86.37512**m with angle **49°**.

3. Discussion

The world record for paper plane maximum distance is 63.1m, which is not as far as the paper plane in my mathematical model. This is I didn't consider the shape and construction of the paper plane. The paper plane may even change its shape during the flight. And the world record is under the indoor condition which may not have any wind speed.

4. Reference

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