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12/17/10

MATH 221

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Newton's Law of Cooling Problem

Newton's law of cooling states that the rate of heat loss from a specific body is proportional to the difference in temperatures between that body and its surroundings. This law can be modeled using a first-order differential equation below:

$$\frac{dT}{dt} = -k(T - T_a).$$

In this equation, dT/dt represents the change in temperature in respect to time, T represents the temperature of the object, T_a is the temperature of the surroundings, and k is a positive constant specific to the object in question. One application of this differential equation in the field of chemical engineering is the determination of the cooling time of a container of liquid required in order to reach a desired temperature. The constant value, k, must be determined experimentally by taking multiple temperate measurements on a set time interval and incorporating the recorded values into the Newton's law of cooling equation. The following problem is an example of how this differential equation can be applied to a real-world scenario.

Problem:

Liquid Benzene is held in a container at 350 K. This container is submerged in a water stream with a constant temperature of 293 K. After 10 minutes, the temperature of the benzene container was measured to be 330 K. What would the temperature be after 15 minutes? 30 minutes? 1 hour? 2 hours? How long would it take for the benzene to reach 320 K? 310 K? 300 K?

In order to answer the questions above, the first step is to find a general solution to the given differential equation. In order to simply the differential equation, y(t) can be substituted for ($T(t) - T_a$) and y_o for ($T_o - T_a$). Differentiating y(t) with respect to t gives the following differential equation shown below:

$$\frac{dy}{dt} = \frac{d}{dt}(T(t) - T_a) = \frac{dT}{dt} - \frac{dT_a}{dt} = \frac{dT}{dt} = -k(T - T_a) = -ky$$

1. Divide both sides of the equation by y: $\left(\frac{1}{y}\right)\frac{dy}{dt} = -k$

2. Integrating both sides: $\ln|y| = -kt + C$

3. Take the exponential of both sides: $y\ t = Ce^{-kt}$ where $C = y_o = y(o)$

The following solution is achieved: $y(t) = y_o e^{-kt}$

By re-substituting T(t) and Ta back into the equation, the following solution is achieved:

$$T(t) = T_a + (T_o - T_a)e^{-kt}$$

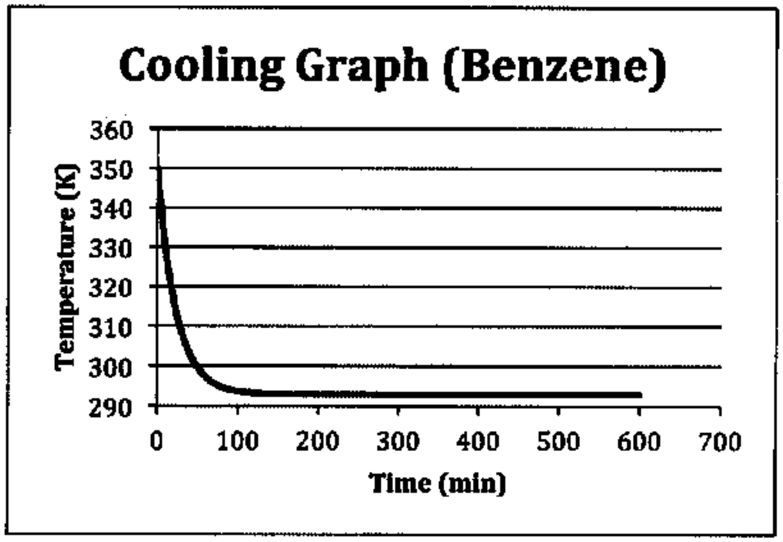
Given the initial conditions,

$$T(0) = T_o = 350K, T_a = 293K, T_o - T_a = 57K$$

$$330K = T(10) = 293K + 57e^{-10k}$$

Solving for k using the given values of T_a, T_o, and T(t), we get $k = 0.0432\ \text{min}^{-1}$.

Below is a graph of the temperature solutions using the above equation (with k = 0.0432):



By plugging in t = 15, 30, 60, and 120 to the equation given above, the following table of temperature solutions was generated:

Time (min)	Temp (K)
0	350.000000
10	330.0049345
15	322.816182
30	308.5965739
60	297.2675985
120	293.3195157

In order to calculate the amount of time elapsed at a given temperature, the given temperate is substituted for T(t) and the unknown variable is t. A sample calculation is included below:

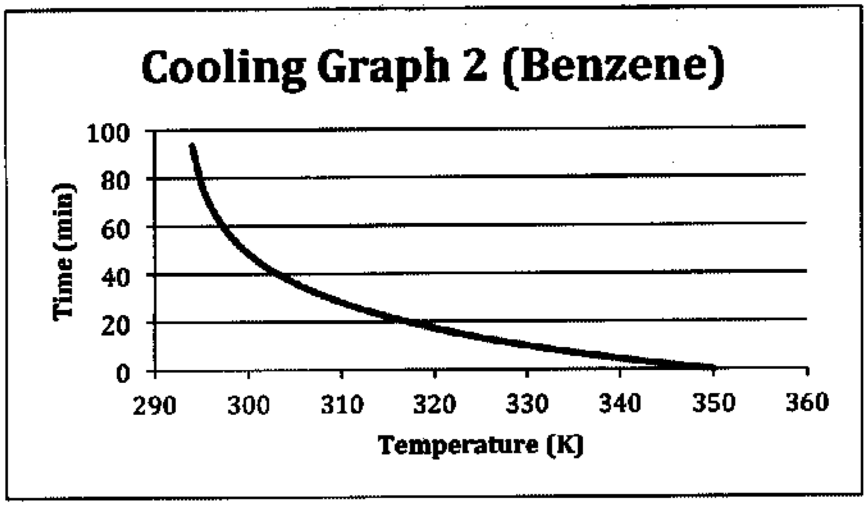
$$320K = T(t) = 293K + 57e^{-0.0432t}$$

$$\frac{-\ln\left(\frac{T(t)-293}{57}\right)}{0.0432} = t$$

Below is a table of the solutions generated by the rearrangement of this equation:

Temp (K)	Time (min)
350	0.00000000
320	17.29662967
310	28.00550749
300	48.54493331

Here is a solution graph of time in respect to temperature:



Based on the tables and graphs above, several conclusions can be drawn about the cooling of an object in surroundings with a constant temperature. Cooling Graph 1 shows a horizontal asymptote at T = 293 K, or T_a. This asymptote verifies the logical assumption that the temperature of the object (T(t)) approaches the temperature of the surroundings (T_a) as time goes to infinity. Looking back at the initial first-order differential equation, this makes sense because the instantaneous rate of change (dT/dt) approaches zero as T(t) approaches Ta. Similarly, Cooling Graph 2 shows a vertical asymptote at T = 293 K. This makes sense because looking at the rearranged time vs. temperature equation, it is clear that as T(t) approaches T_a, time increases infinitely. Basically, this means that the closer T(t) is to T_a, the more time has elapsed since the object began losing heat to its surroundings. Going back and rearranging the initial differential equation gives the following equation: $\frac{dt}{dT} = \frac{-1}{k(T - T_a)}$

This equation suggests that as T(t) approaches T_a, the instantaneous rate of change (dt/dT) approaches negative infinity. In other words, the closer T(t) is to T_a, the more rapidly the amount of time elapsed is decreasing as temperature increases.

As common sense would suggest, at $t = 0$, $T(t) = T_o$ and at T_0, $t = 0$. Also, $T(t)$ achieves a maximum value at $t = 0$. Looking at the initial differential equation, dT/dt achieves a minimum at $t = 0$, which means that at $t = 0$, the temperature is decreasing at the greatest rate. This fact can be verified by examining Cooling Graph 1, where the line appears to have the steepest negative slope at the point $T(0) = 350$.

This model equation was the best for the given problem because it is very simple and heat transfer coefficients can be derived or found experimentally for every system analyzed. Although this often equation provides an accurate approximation, under certain circumstances an accurate formulation may require analysis based on the transient heat transfer equation. This equation can provide a more accurate approximation, but it is much more complicated and for most purposes Newton's law of cooling is sufficient. The equation can be applied to many fields outside of chemical engineering. It can be used for everything from estimating the time of death of a recently discovered corpse to testing the efficiency of an air conditioning unit. In conclusion, this equation was chosen because it provides the best combination of simplicity and accuracy in dealing with heat transfers.